

A note on the exponential sums of the localized divisor functions

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For a fixed integer $k \geq 2$, let us consider $k - 1$ intervals $I_j \subseteq \mathbb{N}$ ($j = 1, \dots, k - 1$) and define $\mathcal{J}_k \stackrel{\text{def}}{=} I_1 \times \dots \times I_{k-1}$. Then, for all $n \in \mathbb{N}$ let us set

$$\Delta_{\mathcal{J}_k}(n) \stackrel{\text{def}}{=} \#\{(n_1, \dots, n_{k-1}) \in \mathcal{J}_k : n_1 \cdots n_{k-1} | n\}.$$

We say that $\Delta_{\mathcal{J}_k}$ is the *divisor function localized on \mathcal{J}_k* . By taking $\mathcal{J}_k = \mathbb{N}^{k-1}$ we recover the standard divisor function $d_k(n)$. Moreover, if all the I_j are intervals of logarithmic length 1, then $\max_{\mathcal{J}_k} \Delta_{\mathcal{J}_k}(n)$ is the *concentration* function introduced by Hooley [Ho] (see also [HT]). Given $\alpha \in [0, 1)$ and $N \in \mathbb{N}$, the exponential sum associated to $\Delta_{\mathcal{J}_k}$ over the integers of $(N, 2N]$ is

$$S_k(\alpha, N) \stackrel{\text{def}}{=} \sum_{n \sim N} \Delta_{\mathcal{J}_k}(n) e(n\alpha),$$

where $n \sim N$ means that $n \in (N, 2N] \cap \mathbb{N}$ and we write $e(\beta)$ for $e^{2\pi i \beta}$. Throughout, the Vinogradov notation $\ll_{k,\varepsilon}$ is synonymous of Landau's $O_{k,\varepsilon}$ ($\varepsilon > 0$ is arbitrarily small and may change at each occurrence).

THEOREM. *For all relatively prime integers a, q with $q > 1$, one has, uniformly in a ,*

$$S_k\left(\frac{a}{q}, N\right) \ll_{k,\varepsilon} (Nq)^\varepsilon \left(\frac{N}{q} + q + N^{1-1/k}\right).$$

Furthermore, this same upper bound applies to $S_k(\alpha, N)$ provided that $\alpha \in (0, 1)$ satisfies $|\alpha - a/q| \leq 1/q^2$.

PROOF. First, let us assume that $\{n \in (N, 2N] \cap \mathbb{N} : \Delta_{\mathcal{J}_k}(n) \neq 0\} \neq \emptyset$ for otherwise the inequality is trivial. Then, we denote $\mathcal{I}_k \stackrel{\text{def}}{=} \mathcal{J}_k \times I_k$ with

$$I_k \stackrel{\text{def}}{=} \{m \in \mathbb{N} : n_1 \cdots n_{k-1} m \sim N \text{ for some } (n_1, \dots, n_{k-1}) \in \mathcal{J}_k\}.$$

Moreover, let us write $S_k(\alpha, N)$ as a multiple exponential sum,

$$S_k(\alpha, N) = \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N}} e(\mathbf{n}_k \alpha),$$

where we set $\vec{n}_k \stackrel{\text{def}}{=} (n_1, \dots, n_k)$ and $\mathbf{n}_k \stackrel{\text{def}}{=} n_1 \cdots n_k$, for brevity. Since it is plain that $\mathbf{n}_k > N$ implies that $n_j > N_k \stackrel{\text{def}}{=} [N^{1/k}]$ for some $j \in \{1, \dots, k\}$ (hereafter $[x]$ is the integer

part of $x \in \mathbb{R}$), we write

$$\begin{aligned}
S_k(\alpha, N) &= \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N \\ n_1 > N_k}} e(\mathbf{n}_k \alpha) + \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N \\ n_1 \leq N_k}} e(\mathbf{n}_k \alpha) \\
&= \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N \\ n_1 > N_k}} e(\mathbf{n}_k \alpha) + \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N \\ n_1 \leq N_k < n_2}} e(\mathbf{n}_k \alpha) + \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N \\ n_1, n_2 \leq N_k}} e(\mathbf{n}_k \alpha) \\
&= \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N \\ n_1 > N_k}} e(\mathbf{n}_k \alpha) + \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N \\ n_1 \leq N_k < n_2}} e(\mathbf{n}_k \alpha) + \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N \\ n_1, n_2 \leq N_k < n_3}} e(\mathbf{n}_k \alpha) + \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N \\ n_1, n_2, n_3 \leq N_k}} e(\mathbf{n}_k \alpha) \\
&= \dots\dots\dots \\
&= \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N \\ n_1 > N_k}} e(\mathbf{n}_k \alpha) + \sum_{j=2}^k \sum_{\substack{\vec{n}_k \in \mathcal{I}_k \\ \mathbf{n}_k \sim N \\ n_1, \dots, n_{j-1} \leq N_k < n_j}} e(\mathbf{n}_k \alpha).
\end{aligned}$$

Of course, it is tacitly understood that if the constraints $n_1, \dots, n_{j-1} \leq N_k$, $N_k < n_j$ are incompatible respectively with $(n_1, \dots, n_{j-1}) \in I_1 \times \dots \times I_{j-1}$, $n_j \in I_j$, then the sum under any of such conditions is meant to be zero. Thus, we have

$$|S_k(\alpha, N)| \leq \sum_{j=1}^k \sum_{\substack{\vec{n}_k^{(j)} \in \mathcal{I}_k^{(j)} \\ \mathbf{n}_k / n_j < 2N/N_k}} \left| \sum_{\substack{n_j \in I_j \\ \mathbf{n}_k \sim N \\ n_j > N_k}} e(\mathbf{n}_k \alpha) \right|,$$

where $\mathcal{I}_k^{(j)} \stackrel{\text{def}}{=} I_1 \times \dots \times I_{j-1} \times I_{j+1} \times \dots \times I_k$ and analogously $\vec{n}_k^{(j)}$ is the $k-1$ dimensional vector obtained by removing the j -th entry from \vec{n}_k . We apply the well-known inequality [D, Ch.25]

$$\left| \sum_{\substack{n_j \in I_j \\ \mathbf{n}_k \sim N \\ n_j > N_k}} e(\mathbf{n}_k \alpha) \right| = \left| \sum_{\substack{n_j \in I_j, n_j > N_k \\ n_j \sim \frac{N n_j}{\mathbf{n}_k}}} e\left(\alpha \frac{\mathbf{n}_k}{n_j} n_j\right) \right| \leq \min\left(\frac{N n_j}{\mathbf{n}_k}, \frac{1}{\|\alpha \mathbf{n}_k / n_j\|}\right),$$

where $\|x\|$ denotes the distance of $x \in \mathbb{R}$ from the integers. Thus, writing $t = \mathbf{n}_k / n_j$,

$$|S_k(\alpha, N)| \leq k \sum_{t < 2N/N_k} d_{k-1}(t) \min\left(\frac{N}{t}, \frac{1}{\|t\alpha\|}\right).$$

Note that $d_1(t) = 1$, while for $k \geq 2$ recall that $d_k(t) \ll_{k,\varepsilon} t^\varepsilon, \forall \varepsilon > 0$. Hence, by taking

$\alpha = a/q$ and denoting with \bar{a} the inverse of $a \pmod{q}$, we get

$$\begin{aligned}
S_k\left(\frac{a}{q}, N\right) &\ll_{k,\varepsilon} N^\varepsilon \left(\frac{N}{q} \sum_{t' < \frac{2N}{qN_k}} \frac{1}{t'} + \sum_{1 \leq r \leq \frac{q}{2}} \frac{q}{r} \sum_{\substack{t < 2N/N_k \\ t \equiv \pm r \bar{a} \pmod{q}}} 1 \right) \\
&\ll_{k,\varepsilon} N^\varepsilon \left(\frac{N}{q} + \sum_{1 \leq r \leq \frac{q}{2}} \frac{q}{r} \left(\frac{N}{qN_k} + 1 \right) \right) \\
&\ll_{k,\varepsilon} (Nq)^\varepsilon \left(\frac{N}{q} + \frac{N}{N_k} + q \right) \\
&\ll_{k,\varepsilon} (Nq)^\varepsilon \left(\frac{N}{q} + q + N^{1-1/k} \right).
\end{aligned}$$

Once $\alpha \in (0, 1)$ is such that $|\alpha - a/q| \leq 1/q^2$, the bound for $S_k(\alpha, N)$ follows by the same calculations to prove (3) in [D, Ch.25]. The Theorem is completely proved. \square

Remark 1. It transpires that the upper bound of $S_k(\alpha, N)$ does not depend on the *localization* of the divisors of $n \in (N, 2N]$. In particular, it holds also for the exponential sum associated to the Hooley's function.

For the divisor function d_k we explicitly state the following.

COROLLARY. *For all relatively prime integers a, q with $q > 1$ we have, uniformly for $\alpha \in [a/q - 1/q^2, a/q + 1/q^2]$ and a ,*

$$\sum_{n \sim N} d_k(n) e(n\alpha) \ll_{k,\varepsilon} (Nq)^\varepsilon \left(\frac{N}{q} + q + N^{1-1/k} \right).$$

Remark 2. Clearly, such an inequality gives some improvement on the trivial bound $N^{1+\varepsilon}$ when $N^{1/k} \ll q \ll N^{1-1/k}$, which indeed yields the estimate $N^{1-1/k+\varepsilon}$. It is under these conditions that such inequalities are most commonly applied.

We are going to use them in our treatment of mean-squares of d_k in short intervals (see [CL1] and [CL2]). In particular, the present investigation on $\Delta_{\mathcal{J}_k}$ has been motivated by our k -folding method [CL1], where we *localize* the divisor function in suitable boxes.

Acknowledgment. The authors wish to thank A. Ivić for useful hints.

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